1.1: Functions

A relation is a collection of ordered pairs.

A function is a relation in which every input has exactly one output.

Two questions:

(1) Input = Age   Output = # of children

(2) Input = Year   Output = # of women in Senate.

Most everything we will see are functions. This is what makes them special.

We write $f: D \to Y$ to mean f is a function with inputs in D and outputs in Y.

D is the domain of f.

For each $x \in D$, we write $f(x) \in Y$ to mean the unique output corresponding to the input x. The collection of all outputs $R = \text{Range} \subseteq Y$.

<u>Ex:</u>

<u>Function</u>	<u>Domain</u>	<u>Range</u>
$y=x^2$	$(-\infty,\infty)$	$[0,\infty)$
$y=1/x$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,0)\cup(0,\infty)$
$y=\sqrt{x}$	$[0,\infty)$	$[0,\infty)$
$y=\sqrt{4-x}$	$(-\infty,4]$	$[0,\infty)$
$y=\sqrt{1-x^2}$	$[-1,1]$	$[0,1]$

The <u>graph</u> of a fcn f is the collection of ordered pairs

$$\{(x,f(x)) : x\in D\}$$

in the Cartesian plane.

Graphically, a fcn must pass the vertical line test

Draw Graphs of all functions Above

<u>Piecewise-Defined Functions</u>

<u>Ex:</u> $|x|=\begin{cases} x, & x\geq 0 \\ -x, & x<0 \end{cases}$ $\qquad f(x)=\begin{cases} -x, & x<0 \\ x^2, & 0\leq x\leq 1 \\ 1, & x>1 \end{cases}$

$\lfloor x\rfloor$ and $\lceil x\rceil$ integer floor and ceiling fcn.

Increasing and Decreasing Functions: (and constant)

1. $f(x_2) > f(x_1)$ whenever $x_1 < x_2$ $\forall\, x_1, x_2 \in I$.
2. $f(x_2) < f(x_1)$ whenever $x_1 < x_2$ $\forall\, x_1, x_2 \in I$.

Even and Odd Functions:

Even if $f(-x) = f(x)$ (even powers of x) symmetric about y-axis

Odd if $f(-x) = -f(x)$ (odd powers of x) symmetric about origin

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Common Functions:

(1) Identity

(2) Linear ($y = Kx$ proportional)

(3) Power Functions

(4) Polynomials

(5) Rational ($y = K/x$ inversely proportional)

(6) Exponential

(7) Algebraic

(8) Trigonometric

(9) Logarithmic

(10) Transcendental (not algebraic)

Catenary, hanging cables.

1.2: Sums, Differences, Products and Quotients

$(f+g)(x) = f(x)+g(x)$ $\quad$ $(f\cdot g)(x) = f(x)\cdot g(x)$

$(f-g)(x) = f(x)-g(x)$ $\quad$ $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$(cf)(x) = cf(x)$.

Ex: $f(x)=\sqrt{x}$ and $g(x)=\sqrt{1-x}$.

Find the 4 above as well as domain and range.

Find general Domain and Range.

Composition: $(f\circ g)(x) = f(g(x))$ Arrow Diagram. Not commutative.

Ex: $f(x)=\sqrt{x}$, $g(x)=x+1$

$f\circ g$, $g\circ f$, $f\circ f$, $g\circ g$. Find Domain and Range.

Is there a general formula?

Consider $\sqrt{x}$ and x^2. Inverse discussion.

Shifting

Vertical – $y=f(x)+k$ (up k)

Horizontal – $y=f(x-h)$ (right h)

Ex: x^2+1, $|x-2|-1$,

$3\sqrt{x}$, $\sqrt{3x}$, $-\sqrt{x}$

Scaling and Reflecting

$y=cf(x)$ (vert. stretch)

$y=\frac{1}{c}f(x)$ (vert. compress)

$y=f(cx)$ (hor. compress)

$y=f(\frac{1}{c}x)$ (hor. stretch)

$y=-f(x)$ (reflect across x-axis)

$y=f(-x)$ (reflect across y-axis)

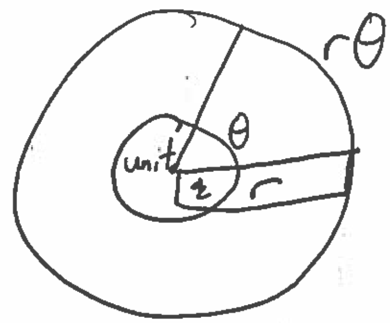
Ex: $f(x) = x^4 - 4x^3 + 10$. Find formulas to

(a) Compress horizontally by 2 and reflect across y-axis. $f(-2x)$

(b) Compress vertically by 2 and reflect across x-axis. $-\frac{1}{2}f(x)$

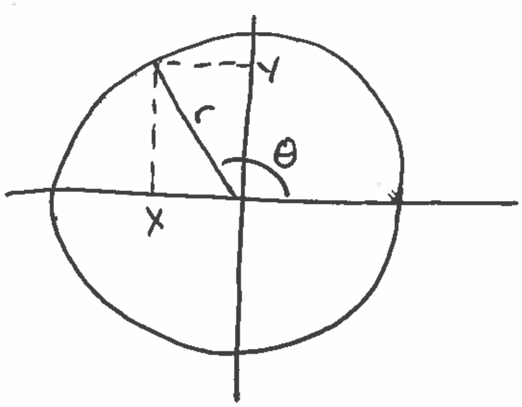
1.3: Trigonometric Functions (Print off unit circles and identities)

Radians and Arc Length



π radians = 180°

We will use radians.



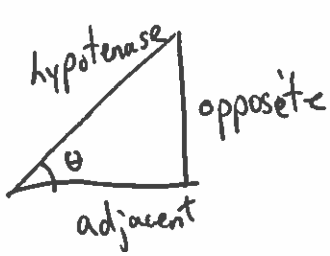
$\sin\theta = \frac{y}{r}$ $\csc\theta = \frac{r}{y}$

$\cos\theta = \frac{x}{r}$ $\sec\theta = \frac{r}{x}$

$\tan\theta = \frac{y}{x}$ $\cot\theta = \frac{x}{y}$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\csc\theta = \frac{1}{\sin\theta}$

$\sec\theta = \frac{1}{\cos\theta}$ $\cot\theta = \frac{1}{\tan\theta}$



SOH CAH TOA

Q: Which trig fcns are odd/even?

<u>Periodic</u>

A fcn f is <u>periodic</u> if there exists $p>0$ such that $f(x+p)=f(x)$ $\forall x \in \mathrm{Dom}(f)$. The smallest such p is called the <u>period</u> of f.

Q: ~~What is the p~~ Which trig. fcns are periodic? What are there periods?

Handout Trigonometric Identities.

<u>Transformations of trig fcns</u>

$$y = a f(b(x+c)) + d$$

sinusoid or general sine fcn

$$f(x) = A\sin\left(\frac{2\pi}{B}(x+C)\right) + D$$

Why not cosinusoid as well?

1.5: Exponential Functions (with base a)

$f(x) = a^x$, $a > 0$. $\text{Dom}(f) = (-\infty, \infty)$ $\text{Ran}(f) = (0, \infty)$

Ex: Starting with \$100 in 2014 and an interest rate of 5.5% compounded annually, write formula for Amount of money x years after 2014.

$$A(x) = 100(1.055)^x.$$

a^n, $a^{1/n}$, $a^{p/q}$, what about $a^{\sqrt{3}}$? (The answer will be more precise as we go along.)

Rules For $a > 0$ and $b > 0$ and all $x, y \in \mathbb{R}$.

(1) $a^x \cdot a^y = a^{x+y}$ (2) $\frac{a^x}{a^y} = a^{x-y}$

(3) $(a^x)^y = a^{xy}$ (4) $a^x \cdot b^x = (ab)^x$

(5) $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$

Ex:

$3^{1.1} \cdot 3^{0.7}$

$\frac{(\sqrt{10})^3}{\sqrt{10}}$

$(5^{\sqrt{2}})^{\sqrt{2}}$

$7^{\pi} \cdot 8^{\pi}$ $\left(\frac{4}{9}\right)^{1/2}$

The natural exponent (Euler's Constant) Discovered by Bernoulli

$$e = 2.71828 1828\ldots$$

The unique exponential fcn with slope 1 on y-axis ($x=0$).

Why not 2 or 10? 10 is a human construct, 2 could be nice, but above prop. is one of many that make it the natural exponent. It makes many calculations much easier, as we will see.

Explain

Exponential growth, decay, compounded continuously, instantaneous rates.

$\underline{Ex}$: Pe^{rt} formula

~~P $100~~ Initial investment of \$100 in 2014, annual interest rate of 5.5% compounded continuously.

$\underline{Ex}$: Laboratory experiments indicate that some atoms emit a part of their mass as radiation, with the remainder forming a ~~new~~ atom of some new element. For example, radioactive Carbon-14 may decay to Nitrogen or

Radium decays to lead.

Radioactive Carbon-14 has instantaneous decay rate of $r = 1.2 \times 10^{-4}$.
How much of initial amount is left after 866 years.

$$y(866) = y_0 e^{-1.2\times10^{-4}(866)} = (0.901) y_0 \Rightarrow 90.1\%.$$

1.6: Inverse Functions and Logarithms

Additive and Multiplicative Identities and Inverses.

A function is invertible if it is 1-1; i.e.

$$f(x_1) = f(x_2) \text{ implies } x_1 = x_2$$

or

$$f(x_1) \neq f(x_2) \text{ whenever } x_1 \neq x_2.$$

$\underline{Ex}$: $\sqrt{x}$, $\sin x$, . Horizontal Line Test

If f is 1-1 on D, we define the inverse on $Ran(f) = R$, by

$$f^{-1}(y) = x \quad \text{if } f(x) = y.$$

$Dom(f^{-1}) = R$ and $Ran(f^{-1}) = D$.

Clear from definition that $(f \circ f^{-1})(x) = x = (f^{-1} \circ f)(x)$.
Hence $f(x) = x$ is the identity fcn.

- Reflection across Identity.
- Rotate 90° counterclockwise and reflect across y-axis.

<u>Ex:</u> Find inverse of $y = \frac{1}{2}x + 1$ and $y = x^2$.
$y = 2x - 2$ and ($y = \sqrt{x}$ if we restrict domain)

<u>Logarithm Functions</u>

$y = \log_a x$ is the logarithm function with base a and is the inverse of $y = a^x$. $(a > 0)(a \neq 1)$.

$$\log_a x = y \quad \text{iff} \quad a^y = x.$$

<u>Natural Log</u>

$\ln x = \log_e x$

<u>Common Log</u>

$\log x = \log_{10} x$.

$\mathrm{Dom}(\log_a x) = (0, \infty)$
$\mathrm{Ran}(\log_a x) = (-\infty, \infty)$

<u>Properties</u> ~~[illegible]~~

(1) $\ln(bx) = \ln(b) + \ln(x)$ (2) $\ln\left(\frac{b}{x}\right) = \ln(b) - \ln(x)$

(3) $\ln\left(\frac{1}{x}\right) = -\ln(x)$ (4) $\ln(x^r) = r\ln(x)$.

<u>Ex:</u> $\ln 4 + \ln \sin x = \ln(4 \sin x)$

$\ln\left(\frac{x+1}{2x-3}\right) = \ln(x+1) - \ln(2x-3)$

$\ln(1/8) = -\ln(8)$

Inverse Function Properties:

(1) $a^{\log_a x} = x$, $\log_a(a^x) = x$

(2) $e^{\ln x} = x$, $\ln(e^x) = x$

(3) In particular, $\log_a(1) = 0$ and $\log_a(a) = 1$.

$a^x = e^{(\ln a) x}$ (show steps)

Change of Base Formula:

$$\log_a(x) = \frac{\ln x}{\ln a} \quad (a > 0, a \neq 1)$$

Pf: $\ln x = \ln(a^{\log_a x}) = \log_a x \cdot \ln a$ ∎

Ex: \$1000 invested earning 5.25% compounded cont.
How long until \$2500? ≈ 17.9 years

Ex: Half-life

polonium-210 has a instantaneous decay rate of 5×10^{-3} (t in days)

≈ 139 days

Inverse Trig Fcns

$\sin^{-1}$ or arcsin $R = [-\pi/2, \pi/2]$

arccos $R = [0, \pi]$

arctan $R = (-\pi/2, \pi/2)$

arccot $R = (0, \pi)$

arcsec $R = (0, \pi/2) \cup (\pi/2, \pi)$

arccsc $R = (-\pi/2, 0) \cup (0, \pi/2)$

Ex: $\arcsin(\sqrt{3}/2) = \pi/3$, $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$